Reduction Formlae, Beta & Gamma Functions

Reduction formulae:-

The term "reduction formula" means a formula which gives the relation between the integral and its simpler form. It reduces a given integral to a known integration form by repeated application of integration by parts.

- Reduction formulae for sinusoidal functions:-

(1) To find a reduction formula for $\int \sin^n x \, dx$, where n is a positive integer ≥ 2. & to evaluate completely $\int_0^{\pi/2} \sin^n x \, dx$.

$$I_n = \int \sin^n x \, dx = \int \underbrace{\sin^{n-1} x}_{u} \cdot \underbrace{\sin x}_{v} \, dx.$$

Using the rule of Integration by parts.

$$I_n = \sin^{n-1} x \cdot (-\cos x) - \int (n-1) \sin^{n-2} x \cos x (-\cos x) dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx.$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

$$I_n = -\sin^{n-1} x \cdot \cos x + (n-1) I_{n-2} - (n-1) I_n.$$

simplify,

$$\therefore I_n + (n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

$$\therefore [1 + (n-1)] I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}.$$

$$\therefore n I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}.$$

$$\therefore I_n = \frac{-1}{n} \sin^{n-1} x \cos x + \frac{(n-1)}{n} I_{n-2}.$$

This is the required reduction formula for $\int \sin^n x$.

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Nao. let, $I_n = \int_0^{\pi/2} \sin^n x\,dx$

Hence substituting the limits,

$$I_n = \left[\frac{-\sin^{n-1}x\cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} I_{n-2}$$

$$I_n = \frac{n-1}{n} I_{n-2} \quad \text{——— (1)}$$

changing n to n-2 in equation (1) successively, we get,

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

$$I_{n-6} = \frac{n-7}{n-6} I_{n-8} \quad \ldots \text{ and so on.}$$

Next, consider two cases,

case(1) :- Let n be a positive even integer.

If n is an even integer, putting n=4 in eqn (1) we get,

$$I_4 = \frac{3}{4} I_2, \text{ Ily, } I_2 = \frac{1}{2} I_0$$

$$I_0 = \int_0^{\pi/2} \sin^0 x \cdot dx = \frac{\pi}{2}$$

$$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

case(2) :- Let n be an odd positive integer.

put n=5 in equation (1),

$$I_5 = \frac{4}{5} I_3 ; \ I_3 = \frac{2}{3} I_1$$

$$I_1 = \int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2} = 1.$$

$$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1.$$

Hence,

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \quad ; \text{ If } n \text{ is even.}$$

$$\& \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1 \, ; \text{ If } n \text{ is odd} \qquad (2)$$

Note:- Using the property,

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\therefore \int_0^{\pi/2} \sin^n(x)\,dx = \int_0^{\pi/2} \sin^n\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi/2} \cos^n x\,dx$$

from eqn (2)

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \, ; \text{ if } n \text{ is even}$$

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1 \, ; \text{ if } n \text{ is odd.}$$

Additional Results:-

(1) $\int_0^{\pi} \sin^n x\,dx = 2\int_0^{\pi/2} \sin^n x\,dx$, $\forall n$ integral values of n.

(2) $\int_0^{\pi} \cos^n x\,dx = 2\int_0^{\pi/2} \cos^n x\,dx$; if n is an even integer

$= 0$; if n is an odd integer.

(3) $\int_0^{2\pi} \sin^n x\,dx = 4\int_0^{\pi/2} \sin^n x\,dx$; if n is an even integer

$= 0$; if n is an odd integer.

(4) $\int_0^{2\pi} \cos^n x\,dx = 4\int_0^{\pi/2} \cos^n x\,dx$; if n is an even integer

$= 0$; if n is an odd integer.

(2) To find a reduction formula for $\int \sin^m x \cos^n x\,dx$ where, m & n are positive integers ≥ 2 and to completely evaluate $\int_0^{\pi/2} \sin^m x \cdot \cos^n x\,dx$.

Let $I_{m,n} = \int \sin^m x \cdot \cos^n x\,dx$

$$= \int \sin^m x \cdot \cos^{n-1} x \cdot \cos x\,dx$$

$$= \int \underbrace{\cos^{n-1} x}_{u} \cdot \underbrace{(\sin^m x \cdot \cos x)}_{v} dx$$

$\because \int \sin^m x \cos x\,dx = \frac{\sin^{m+1} x}{m+1}$ $\left\{ \because \int [f(x)]^m f'(x)dx = \frac{[f(x)]^{m+1}}{m+1} \right.$

Now, applying integration by parts,

$$I_{m,n} = \cos^{n-1} x \cdot \frac{\sin^{m+1} x}{m+1} - \int (n-1)\cos^{n-2} x(-\sin x)\frac{\sin^{m+1} x}{m+1} dx$$

$$\therefore I_{m,n} = \frac{\cos^{n-1} x \cdot \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2} x \cdot \cos^{n-2} x\,dx.$$

$$= \frac{\cos^{n-1} x \cdot \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cdot \sin^2 x \cdot \cos^{n-2} x\,dx$$

$$= \frac{\cos^{n-1} x \cdot \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cdot (1-\cos^2 x)\cos^{n-2} x\,dx$$

$$= \frac{\cos^{n-1} x \cdot \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x\,dx - \frac{n-1}{m+1}\int \sin^m x \cdot \cos^n x\,dx$$

$$\therefore I_{m,n} = \frac{\cos^{n-1} x \cdot \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}.$$

$$\therefore I_{m,n} + \frac{n-1}{m+1} I_{m,n} = \frac{\cos^{n-1} x \cdot \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$\therefore I_{m,n}\left(\frac{m+1+n-1}{m+1}\right) = \frac{\cos^{n-1} x \cdot \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$\therefore I_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

$$\int \sin^m x \cdot \cos^n x\,dx = \frac{\cos^{n-1}x \cdot \sin^{m+1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cdot \cos^{n-2}x\,dx \quad \text{--- (1)}$$

which is required reduction formula.

from eqⁿ (1)

$$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \left[\frac{\cos^{n-1}x \cdot \sin^{m+1}x}{m+n}\right]_0^{\pi/2} + \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cdot \cos^{n-2}x\,dx$$

$$= 0 + \frac{n-1}{m+n} I_{m,n-2}.$$

$$\therefore \boxed{I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}} \quad \text{--- (2)}$$

Replacing n by n-2 in eqⁿ (2) successively, we get.

$$I_{m,n-2} = \frac{n-3}{m+n-2} I_{m,n-4}$$

$$I_{m,n-4} = \frac{n-5}{m+n-4} I_{m,n-6} \ldots\ldots$$

$$\therefore I_{m,n} = \left(\frac{n-1}{m+n}\right)\cdot\left(\frac{n-3}{m+n-2}\right)\cdot\left(\frac{n-5}{m+n-4}\right). I_{m,n-6} \ldots \text{ and so on.}$$

case(i):- let n be an even positive integer.

$$I_{m,4} = \frac{3}{m+4} I_{m,2} = \frac{3}{m+4}\cdot\frac{1}{m+2}. I_{m,0}$$

$$I_{m,n} = \frac{n-1}{m+n}\cdot\frac{n-3}{m+n-2}\cdot\frac{n-5}{m+n-4}\ldots\ldots\frac{3}{m+4}\,\frac{1}{m+2}. I_{m,0}$$

&

$$I_{m,0} = \int_0^{\pi/2} \sin^m x \cdot \cos^0 x\,dx = \int_0^{\pi/2} \sin^m x\,dx.$$

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$$\therefore I_{m,0} = \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdot \frac{m-5}{m-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \text{ if } m = \text{even}$$

$$= \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdot \frac{m-5}{m-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1, \text{ if } m = \text{odd}.$$

$\therefore$ If both m & n are even integers.

$$\boxed{\therefore I_{m,n} = \frac{\{(n-1)(n-3)\cdots 3\cdot 1\}\cdot\{(m-1)\cdot(m-3)\cdots 3\cdot 1\}}{\{(m+n)(m+n-2)\cdots 4\cdot 2\}} \cdot \frac{\pi}{2} \quad \cdots m, n \text{ both even}}$$

whereas, if m is odd and n is even, then

$$I_{m,n} = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

$$\boxed{\therefore I_{m,n} = \frac{\{(n-1)(n-3)\cdots 3\cdot 1\}\cdot\{(m-1)(m-3)\cdots 4\cdot 2\}}{\{(m+n)(m+n-2)\cdots 5\cdot 3\cdot 1\}} \; ; m\text{-odd}, n\text{-even}}$$

Case (ii) Let n be an odd integer.
from eqn (2),

$$I_{m,5} = \frac{4}{m+5} I_{m,3} = \frac{4}{m+5} \cdot \frac{2}{m+3} \cdot I_{m,1}$$

$$I_{m,n} = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \cdots \frac{4}{m+5} \cdot \frac{2}{m+3} \cdot I_{m,1}$$

$$\& \; I_{m,1} = \int_0^{\pi/2} \sin^m x \cdot \cos x \, dx = \left[\frac{\sin^{m+1} x}{m+1}\right]_0^{\pi/2} = \frac{1}{m+1}$$

$\therefore$ If n is odd and m may be even or odd, then,

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \left[\frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{2}{m+3}\right] \frac{1}{m+1}$$

This is also written as,

$$\boxed{\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx = \left[\frac{\{(n-1)(n-3)\cdots 4\cdot 2\}\{(m-1)(m-3)\cdots 3\cdot 1\}}{(m+n)(m+n-2)\cdots(m+3)(m+1)(m-1)(m-3)\cdots 3\cdot 1}\right]}$$

$m = \text{even}; n = \text{odd}$

$$\boxed{\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx = \left[\frac{\{(n-1)(n-3)\cdots 4\cdot 2\}\{(m-1)(m-3)\cdots 4\cdot 2\}}{(m+n)(m+n-2)\cdots(m+3)(m+1)(m-1)(m-3)\cdots 4\cdot 2}\right]}$$

if $m = \text{odd}; n = \text{odd}$.

combining all the cases, we get

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\{(m-1)(m-3)\ldots 2 \text{ or } 1\}\{(n-1)(n-3)\ldots 2\}}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1} \times P.$$

where $P = \frac{\pi}{2}$, if m and n are both even.

$= 1$; for all other values of m and n.

e.g. ① $\int_0^{\pi/2} \sin^6 x \cdot \cos^4 x \, dx = \frac{(5.3.1)(3.1)}{10.8.6.4.2} \times \frac{\pi}{2} = \frac{3\pi}{512}$

② $\int_0^{\pi/2} \sin^5 x \cdot \cos^6 x \, dx = \frac{(4.2)(5.3.1)}{11.9.7.5.3.1} \times 1 = \frac{8}{693}$

Additional Results:-

(1) $\int_0^{\pi/2} \sin^P x \cos x \, dx = \frac{1}{P+1} = \int_0^{\pi/2} \cos^P x \cdot \sin x \, dx$

(2) $\int_0^{\pi} \sin^m x \cdot \cos^n x \, dx = 2\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx$; if n-even, m=even or odd.

$= 0$, if n=odd, m-even or odd.

(3) $\int_0^{2\pi} \sin^m x \cdot \cos^n x \, dx = 4\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx$; m, n-even

$= 0$; otherwise

(3) Reduction formula for $\int \tan^n x \, dx$.

Let $I_n = \int \tan^n x \, dx = \int \tan^{n-2} x \cdot \tan^2 x \, dx$

$= \int \tan^{n-2} x \cdot (\sec^2 x - 1) dx$

$= \int \tan^{n-2} x \cdot \sec^2 x \, dx - \int \tan^{n-2} x \, dx$

$$I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$$

$$\int \tan^n x\, dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\, dx$$

which is the required reduction formula.

(4) Reduction formula for $\int \sec^n x\, dx$.

Let $I_n = \int \sec^n x\, dx$.

$$= \int \sec^{n-2}x \cdot \sec^2 x\, dx.$$

Using Integration by parts.

$$I_n = \sec^{n-2}x \tan x - \int (n-2) \sec^{n-3}x(\sec x \tan x)\tan x\, dx.$$

$$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2}x(\sec^2 x - 1)dx.$$

$$= \sec^{n-2}x \tan x - (n-2)\int \sec^n x\, dx + (n-2)\int \sec^{n-2}x\, dx$$

$$I_n = \sec^{n-2}x \tan x - (n-2)I_n + (n-2)I_{n-2}$$

$$(n-1)I_n = \sec^{n-2}x \tan x + (n-2)I_{n-2}$$

$$I_n = \frac{\sec^{n-2}x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$\therefore \int \sec^n x\, dx = \frac{\sec^{n-2}x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\, dx.$$

which is required reduction formula.

Que. ① Evaluate $\int_0^{2a} x\sqrt{2ax - x^2}\, dx$.

Sol^n: Let $I = \int_0^{2a} x\sqrt{2ax - x^2}\, dx = \int_0^{2a} x \cdot \sqrt{x}\sqrt{2a - x}\, dx.$

$$= \int_0^{2a} x^{3/2} \cdot (2a-x)^{1/2} dx$$

put $x = 2a\sin^2\theta$, $dx = 4a\sin\theta \cdot \cos\theta \, d\theta$.

x	0	$2a$
θ	0	$\pi/2$

$$I = \int_0^{\pi/2} (2a)^{3/2} \cdot \sin^3\theta \left(2a - 2a\sin^2\theta\right)^{1/2} \cdot 4a\sin\theta\cos\theta \, d\theta$$

$$= 16a^3 \int_0^{\pi/2} \sin^4\theta \cos^2\theta \, d\theta$$

$$= 16a^3 \cdot \frac{(3 \cdot 1)(1)}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$\boxed{I = \frac{\pi a^3}{2}}$$

(2) Evaluate $\int_0^{\pi} x\sin^7 x \cdot \cos^4 x \, dx$.

Soln:- Let $I = \int_0^{\pi} x\sin^7 x \cdot \cos^4 x \, dx$ ——— (1)

$$= \int_0^{\pi} (\pi - x)\sin^7(\pi - x) \cdot \cos^4(\pi - x) \, dx$$

$$\left\{ \because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \right.$$

Also, $\sin(\pi - x) = \sin x$, $\cos(\pi - x) = -\cos x$

$$I = \int_0^{\pi} (\pi - x)\sin^7 x \cos^4 x \, dx \quad \text{——— (2)}$$

Adding (1) & (2)

$$2I = \int_0^{\pi} \pi \sin^7 x \cdot \cos^4 x \, dx$$

$$= \pi \cdot 2 \int_0^{\pi} \sin^7 x \cdot \cos^4 x \, dx$$

$$= \pi \cdot \frac{6 \cdot 4 \cdot 2 \cdot 3 \cdot 1}{11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}$$

$$\boxed{I = \frac{16\pi}{1155}}$$

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(3) If $I_n = \int_0^{\pi/4} \frac{\sin(2n-1)x}{\sin x}\,dx$ then prove that,
$n(I_{n+1} - I_n) = \sin\frac{n\pi}{2}$ and hence, find I_3.

Sol^n:-

Given $I_n = \int_0^{\pi/4} \frac{\sin(2n-1)x}{\sin x}\,dx$

$\therefore I_{n+1} = \int_0^{\pi/4} \frac{\sin(2n+1)x}{\sin x}\,dx$

$\therefore I_{n+1} - I_n = \int_0^{\pi/4} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x}\,dx$

$= \int_0^{\pi/4} \frac{2\cos 2nx \sin x}{\sin x}\,dx$ $\quad \left\{ \because \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2} \right.$

$= 2\left[\frac{\sin 2nx}{2n}\right]_0^{\pi/4}$

$= \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$

$\therefore n(I_{n+1} - I_n) = \sin\frac{n\pi}{2}$

put $n=2$, $\quad 2(I_3 - I_2) = 0$

put $n=1$, $\quad I_2 - I_1 = 1$.

& $\quad I_1 = \int_0^{\pi/4} \frac{\sin x}{\sin x}\,dx = \frac{\pi}{4}$

$I_3 = I_2 = 1 + I_1 = 1 + \frac{\pi}{4}$

$\therefore \boxed{I_3 = 1 + \frac{\pi}{4}}$

Que:- If $I_n = \int_0^{\infty} e^{-x}\sin^n x\,dx$, obtain the relation between I_n & I_{n-2} and hence, find I_4.

Sol:- Given $I_n = \int_0^\infty e^{-x} \sin^n x \, dx$

Using Integration by parts,

$$= \left[\sin^n x \cdot (-e^{-x}) \right]_0^\infty - \int_0^\infty n \sin^{n-1} x \cdot \cos x (-e^{-x}) dx$$

$$= 0 + n \int_0^\infty e^{-x} (\sin^{n-1} x \cos x) dx.$$

$$= \left[n \cdot \sin^{n-1} x \cos x \cdot (-e^{-x}) \right]_0^\infty -$$

$$- n \int_0^\infty \left[\sin^{n-1} x (-\sin x) + (n-1) \sin^{n-2} x \cos^2 x \right] (-e^{-x}) dx$$

$$= 0 + n \int_0^\infty \left[-\sin^n x \, e^{-x} + (n-1) \sin^{n-2} x (1 - \sin^2 x) \cdot e^{-x} \right] dx$$

$$= -n \int_0^\infty e^{-x} \cdot \sin^n x \, dx + n(n-1) \int_0^\infty e^{-x} \sin^{n-2} x \, dx -$$

$$n(n-1) \int_0^\infty e^{-x} \sin^n x \, dx.$$

$$= -n^2 \int_0^\infty e^{-x} \sin^n x \, dx + n(n-1) I_{n-2}$$

$\therefore I_n = -n^2 I_n + n(n-1) I_{n-2} \Rightarrow I_n (n^2+1) = n(n-1) I_{n-2}$

$$\boxed{I_n = \frac{n(n-1)}{n^2+1} I_{n-2}.}$$

put $n=4$, $I_4 = \frac{4 \cdot (3)}{17} \cdot I_2$

put $n=2$, $I_2 = \frac{2(1)}{5} \cdot I_0$

& $I_0 = \int_0^\infty e^{-x} dx$

$\therefore I_4 = \frac{12}{17} \cdot \frac{2}{5} \cdot I_0 = \frac{12}{17} \cdot \frac{2}{5} \cdot \left[-e^{-x} \right]_0^\infty = \frac{24}{85} [0+1]$

$$\boxed{I_4 = \frac{24}{85}}$$

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Gamma function:-

Consider the definite integral $\int_0^\infty e^{-x} x^{n-1} dx$.

It is denoted by the symbol Γn, and is called as Gamma function of n. Thus,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx, \ (n>0)$$

Gamma function is called as Euler's integral of second kind.

Properties of Gamma functions:-

(1) $$\Gamma n = 2\int_0^\infty e^{-x^2} \cdot x^{2n-1} dx$$

$\Rightarrow$ we have, $\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1} dx$

put $x = t^2$, $dx = 2t\,dt$

x	0	∞
t	0	∞

$$= \int_0^\infty e^{-t^2} \cdot t^{2n-2}\, 2t\,dt$$

$$= 2\int_0^\infty e^{-t^2} \cdot t^{2n-1} dt$$

$$\boxed{\Gamma n = 2\int_0^\infty e^{-x^2} \cdot x^{2n-1} dx}$$

(2) $\Gamma 1 = 1$

$\Rightarrow$ by defn. $\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1} dx$

put $n=1$, $\Gamma 1 = \int_0^\infty e^{-x} \cdot x^0 dx$

$$= \left[-e^{-x}\right]_0^\infty$$

$$= (-e^{-\infty} + e^0) = 0+1 = 1$$

$$\therefore \boxed{\Gamma 1 = 1}$$

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(3) Reduction formula for Gamma functions:-

$$\Gamma(n+1) = n\Gamma n.$$

⇒ By defⁿ.

$$\Gamma n = \int_0^\infty e^{-x}.x^{n-1}dx$$

Replace n by n+1,

$$\Gamma n+1 = \int_0^\infty e^{-x}.x^n\,dx.$$

Now, integrating by parts.

$$\Gamma n+1 = \left\{x^n(-e^{-x})\right\}_0^\infty - \int_0^\infty nx^{n-1}(-e^{-x})\,dx.$$

Now, $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$, Also, if $n>0$, $\frac{x^n}{e^x} = 0$, for $x=0$

$$\therefore \left[\frac{x^n}{e^x}\right]_0^\infty = 0$$

$$\Gamma n+1 = 0 + n\int_0^\infty e^{-x}x^{n-1}\,dx$$

$$\Gamma n+1 = n\Gamma n$$

If n is a positive integer,

$$\Gamma n+1 = n(n-1)\Gamma(n-1) \qquad \because \Gamma n = (n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$= n(n-1)(n-2)(n-3)(n-4)\ldots 3.2.1.\Gamma 1$$

$$= n(n-1)(n-2)(n-3)\ldots 3.2.1 \qquad \because \{\Gamma 1 = 1\}$$

$\therefore \Gamma n+1 = n!$, if n is a positive integer

Hence, $\Gamma n+1 = n\Gamma n$, in general

$= n!$, if n is positive integer.

(4) $\Gamma 0 = \infty$ ⇒ we know that $\Gamma n+1 = n\Gamma n$.

$$\therefore \Gamma n = \frac{\Gamma n+1}{n}$$

put $n=0$ $\therefore \Gamma 0 = \frac{\Gamma 1}{0} = \frac{0!}{0} = \frac{1}{0} = \infty.$

(5) $\Gamma\frac{1}{2} = \sqrt{\pi}$

(6) since $\Gamma n+1 = n!$

e.g. $\Gamma 6 = 5!$, $\Gamma 8 = 7!$, $\Gamma 2 = 1! = 1$

$$\Gamma\frac{5}{2} = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\frac{3}{2} = \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

(7) For negative fraction n, we use $\Gamma n = \frac{\Gamma n+1}{n}$

$$\Gamma\frac{-5}{3} = \left(\frac{-3}{5}\right)\Gamma\left(\frac{-2}{3}\right) = \left(\frac{-3}{5}\right)\left(\frac{-3}{2}\right)\Gamma\frac{1}{3} = \frac{9}{10}\Gamma\frac{1}{3}$$

Transformation of Gamma function:-

(1) we know that; $\Gamma n = \int_0^\infty e^{-x} . x^{n-1} dx$

put $x = ky \Rightarrow dx = k\,dy$

x	0	∞
y	0	∞.

$$= \int_0^\infty e^{-ky} k^{n-1} . y^{n-1} k\,dy$$

$$\Gamma n = k^n \int_0^\infty e^{-ky} . y^{n-1} dy$$

$$\therefore \boxed{\int_0^\infty e^{-ky} y^{n-1} dy = \frac{\Gamma n}{k^n}}$$

(2) consider $\Gamma n = \int_0^\infty e^{-x} . x^{n-1} dx$.

put $x^n = y \Rightarrow x = y^{1/n}$.

$\Rightarrow \therefore n x^{n-1} dx = dy$

$\Rightarrow x^{n-1} dx = dy/n$.

x	0	∞
y	0	∞

$$\therefore \Gamma n = \int_0^\infty e^{-y^{1/n}} . \frac{dy}{n}$$

$$\therefore \boxed{\int_0^\infty e^{-y^{1/n}} dy = n\Gamma n = \Gamma n+1}.$$

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put $n=\frac{1}{2}$, $\int_0^\infty e^{-y^2}dy = \frac{1}{2}\Gamma\frac{1}{2}$

but $\int_0^\infty e^{-y^2}dy = \frac{1}{2}\sqrt{\pi}$

$\therefore \frac{1}{2}\sqrt{\pi} = \frac{1}{2}\Gamma\frac{1}{2}$

$\Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$

(3) consider, $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$

put $e^{-x}=y$ $\Rightarrow e^{x}=\frac{1}{y}$ $\Rightarrow x=\log\frac{1}{y}$

$\therefore -e^{-x}dx = dy$

x	0	∞
y	1	0

$\therefore \Gamma n = \int_1^0 \left(\log\frac{1}{y}\right)^{n-1}(-dy)$

$$\boxed{\Gamma n = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1}dy}$$

Additional Result:-

$\Gamma p\,\Gamma 1-p = \frac{\pi}{\sin p\pi}$ if $0<p<1$.

e.g. $\Gamma\frac{1}{4}\Gamma\frac{3}{4} = \Gamma\frac{1}{4}\Gamma 1-\frac{1}{4}$ Let $p=\frac{1}{4}<1$

$= \frac{\pi}{\sin\left(\frac{1}{4}\pi\right)} = \frac{\pi}{1/\sqrt{2}} = \sqrt{2}\pi$.

(2) $\Gamma\frac{1}{3}\Gamma\frac{2}{3} = \Gamma\frac{1}{3}\Gamma 1-\frac{1}{3}$, Let $p=\frac{1}{3}<1$

$= \frac{\pi}{\sin\frac{\pi}{3}} = \frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{\sqrt{3}}$

(1) Evaluate $\int_0^\infty \sqrt[4]{x}\cdot e^{-\sqrt{x}}\,dx$ (Reducible to std form)

Soln:- let $I = \int_0^\infty x^{1/4}\cdot e^{-\sqrt{x}}\,dx$

put $\sqrt{x} = t$ or $x = t^2$

$dx = 2t\,dt$

x	0	∞
t	0	∞

$$= \int_0^\infty t^{1/2}\cdot e^{-t}\, 2t\,dt$$

$$= 2\int_0^\infty e^{-t}\cdot t^{3/2}\,dt$$

$$= 2\,\Gamma\left(\tfrac{5}{2}\right) = 2\,\Gamma\left(\tfrac{3}{2}+1\right) = 2\cdot\tfrac{3}{2}\,\Gamma\left(\tfrac{1}{2}+1\right) = 2\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right)$$

$$= 2\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} = \frac{3\sqrt{\pi}}{2}$$

(2) Evaluate $\int_0^\infty \frac{x^2}{3^{x^2}}\,dx$ (Variable pbms involving $(\text{const})^{(x^2)}$ Hint :- put $a^{x} = e^{t}$)

Soln:- Let $I = \int_0^\infty \frac{x^2}{3^{x^2}}\,dx$

put $3^{x^2} = e^{t}$

x	0	∞
t	0	∞

$\therefore x^2 \log 3 = t$

$\therefore x^2 = \frac{t}{\log 3}$

$x = \frac{t^{1/2}}{\sqrt{\log 3}}$ $\therefore dx = \frac{\frac{1}{2}t^{-1/2}}{\sqrt{\log 3}}\,dt$

$\therefore$ The Integral becomes

$$I = \int_0^\infty \frac{t}{\log 3}\cdot\frac{1}{e^t}\cdot\frac{\frac{1}{2}t^{-1/2}}{\sqrt{\log 3}}\,dt$$

$$= \frac{1}{2}\cdot\frac{1}{(\log 3)^{3/2}}\int_0^\infty e^{-t}\cdot t^{1/2}\,dt$$

$$= \frac{1}{2(\log 3)^{3/2}}\,\Gamma\left(\tfrac{3}{2}\right) = \frac{1}{2(\log 3)^{3/2}}\cdot\frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{4(\log 3)^{3/2}}$$

Hint :- (Pbms involving $\log x$. put $\Rightarrow \log \frac{1}{x} = t$ i.e $-\log x = t$, $\log x = -t$)

(3) Evaluate $\int_0^1 (x \log x)^4 \, dx$

Sol^n:- Let $I = \int_0^1 (x \log x)^4 \, dx$

put $\log x = -t$

$\Rightarrow x = e^{-t}$

$\Rightarrow dx = -e^{-t} \, dt$

x	0	1
t	∞	0

from given integral,

$$I = \int_{\infty}^{0} (e^{-t})^4 \cdot (-t)^4 (-e^{-t} \, dt)$$

$$= \int_{\infty}^{0} t^4 \cdot e^{-5t} \, dt$$

put $5t = u \Rightarrow t = \frac{u}{5} \Rightarrow dt = \frac{du}{5}$

$$= \int_0^{\infty} \left(\frac{u}{5}\right)^4 \cdot e^{-u} \cdot \frac{du}{5}$$

$$= \frac{1}{5^5} \int_0^{\infty} u^4 \cdot e^{-u} \, du$$

$$\boxed{I = \frac{\Gamma 5}{5^5} = \frac{4!}{5^5}}$$

(4) show that $\frac{2^n \Gamma(n+\frac{1}{2})}{\sqrt{\pi}} = 1.3.5 \ldots (2n-1)$ (Use $\Gamma(n+1) = n\Gamma n$)

Sol^n:- consider $\Gamma\left(n+\frac{1}{2}\right)$, Apply $\Gamma(n+1) = n\Gamma n$ repeatedly, we get,

$$\Gamma\left(n+\frac{1}{2}\right) = \Gamma\left(\left(n-\frac{1}{2}\right)+1\right)$$

$$= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\Gamma\left(n-\frac{5}{2}\right) \ldots \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}$$

$$= \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdot \frac{2n-5}{2} \cdots \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}$$

$$= \frac{(2n-1)(2n-3)(2n-5)\cdots 3 \cdot 1 \cdot \sqrt{\pi}}{2^n}$$

$$\therefore \frac{2^n \, \Gamma(n+\tfrac{1}{2})}{\sqrt{\pi}} = 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

Beta function:-

Def^n:- A Beta function of m, n is denoted by, $\beta(m,n)$ & defined as

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad (m>0,\ n>0)}$$

The beta function is also called as Euler's integral of first kind.

Note:- Beta function of negative numbers is not defined.

properties of Beta function:-

① $\beta(m,n) = \beta(n,m)$.

$$\Rightarrow \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

put $1-x = t \Rightarrow -dx = dt$

x	0	1
t	1	0

$$= \int_1^0 (1-t)^{m-1} \cdot t^{n-1}\,(-dt)$$

$$= \int_0^1 t^{n-1} \cdot (1-t)^{m-1}\,dt$$

$$= \beta(n,m)$$

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(2) $\int_0^1 x^m (1-x)^n \, dx = B(m+1, n+1)$

(3) $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$

$\Rightarrow B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

put $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta \, d\theta$

x	0	1
θ	0	$\pi/2$

$$= \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta \, d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$$

we consider this as a defn of beta function

further Let $2m-1 = p$, $2n-1 = q$

$\therefore m = \frac{p+1}{2}$, $n = \frac{q+1}{2}$

$$\therefore B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta$$

$$\boxed{\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$$

(4) Alternating defination.

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx.$$

$\Rightarrow B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

put $x = \frac{t}{1+t}$ i.e $x(1+t) = t$

$\Rightarrow x + xt = t$

$\Rightarrow x = t - xt$

$\Rightarrow t = \frac{x}{1-x}$

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(5) Relation between beta + Gamma functions:-

$$\boxed{\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}}$$

(6) $\Gamma\frac{1}{2} = \sqrt{\pi}$.

we know, $\int_0^{\pi/2} \sin^p\theta . \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$$= \frac{1}{2} \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$$

put $p = q = 0$

$$\int_0^{\pi/2} d\theta = \frac{1}{2} \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 1} \Rightarrow \frac{\pi}{2} = \frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2$$

$$\therefore \Gamma\frac{1}{2} = \sqrt{\pi}.$$

Duplication formula of Gamma function:-

$$\Gamma m \, \Gamma m+\frac{1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m.$$

Additional Result:-

$$\Gamma p \, \Gamma 1-p = \frac{\pi}{\sin p\pi}, \quad 0 < p < 1.$$

EX:- Evaluate $\int_3^7 (x-3)^{1/4}(7-x)^{1/4}\,dx$

Sol:- $I = \int_3^7 (x-3)^{1/4}(7-x)^{1/4}\,dx$

Hint:- problems involving general form
$\int_a^b (x-a)^l (b-x)^m\,dx$.
- In this type use std substitution
$x-a = (b-a)t$.

$\therefore$ put $x-3=4t \Rightarrow dx = 4\,dt$.

x	3	7
t	0	1

$$\therefore I = \int_0^1 (4t)^{1/4}\cdot(7-4t-3)^{1/4}\,dt$$

$$= \int_0^1 4^{1/4}\cdot t^{1/4}\cdot[4(1-t)]^{1/4}\cdot 4\,dt$$

$$= 4^{3/2}\int_0^1 t^{1/4}(1-t)^{1/4}\,dt$$

$$= 8\,\beta\left(\frac{5}{4},\frac{5}{4}\right)$$

$$= 8\,\frac{\Gamma\frac{5}{4}\,\Gamma\frac{5}{4}}{\Gamma\frac{5}{2}}$$

$$= 8\,\frac{\frac{1}{4}\Gamma\frac{1}{4}\cdot\frac{1}{4}\Gamma\frac{1}{4}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$= 8\,\frac{\left(\frac{1}{4}\Gamma\frac{1}{4}\right)^2}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{2}{3\sqrt{\pi}}\left(\Gamma\frac{1}{4}\right)^2$$

Ex ① prove that $\Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \pi\sqrt{2}$

By using duplication formulae,

$$\Gamma m \,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma 2m}{2^{2m-1}}.$$

$$\Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \Gamma\frac{1}{4}\,\Gamma\left(\frac{1}{4}+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma 2\left(\frac{1}{4}\right)}{2^{2(1/4)-1}} = \frac{\sqrt{\pi}\,\Gamma\frac{1}{2}}{2^{-1/2}}.$$

$$(\because m = 1/4)$$

$$= \sqrt{2}.\sqrt{\pi}\sqrt{\pi} = \pi\sqrt{2}.$$

$$\therefore \Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \pi\sqrt{2}.$$

or

using result $\Gamma p\,\Gamma 1-p = \frac{\pi}{\sin p\pi}$, $0<p<1$, $p=\frac{1}{4}$

$$\therefore \Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\frac{1}{\sqrt{2}}} = \pi\sqrt{2}$$

Ex ② prove that $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

Sol^n:-

$$I = \int_0^\infty \frac{dx}{1+x^4}$$

put $x^2 = \tan\theta$

$x = \sqrt{\tan\theta}$

x	0	∞
θ	0	$\pi/2$

$$dx = \frac{1}{2}\tan^{-1/2}\theta.\sec^2\theta.d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\tan^{-1/2}\theta\, d\theta. = \frac{1}{2}\int_0^{\pi/2}\sin^{-1/2}\theta.\cos^{1/2}\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\,\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} = \frac{1}{4}\,\frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{\Gamma 1} = \frac{1}{4}\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{4}\,\frac{\pi}{\sin \pi/4} = \frac{\pi}{2\sqrt{2}}$$

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Differentiation under the integral sign & ERROR FUNCTION.

When a definite integral $I = \int_a^b f(x,\alpha)\,dx$, which is to be integrated w.r.t. variable x and contains only one parameter α, then we diffn I w.r.t. parameter α, by using DUIS.

$I = \int_a^b f(x,\alpha)\,dx$, Here, α - parameter
x - variable of integratn
a & b - limit of integration (may be const or functn of α)

Rule (1) Integrals with const limits i.e a and b are constants.

If $I(\alpha) = \int_a^b F(x,\alpha)\,dx$, where a & b are const. then

$$\frac{dI}{d\alpha} = \int_a^b \frac{\partial}{\partial\alpha} F(x,\alpha)\,dx.$$

i.e. if a & b are const then derivative w.r.t. parameter α outside the definite integral becomes partial derivative inside the integral.

Ex ① Show that $\int_0^1 \frac{x^a - 1}{\log x}\,dx = \log(1+a)$; $(a \geq 0)$

$\Rightarrow$ Consider the integral as $I(a)$.

$$I(a) = \int_0^1 \frac{x^a - 1}{\log x}\,dx$$

differentiate both sides w.r.t. a.

$$I'(a) = \frac{d}{da}\int_0^1 \frac{x^a - 1}{\log x}\,dx$$

Apply DUIS, $I'(a) = \int_0^1 \frac{\partial}{\partial a} \frac{x^a - 1}{\log x} dx$

$$\Rightarrow I'(a) = \int_0^1 \frac{x^a \log x}{\log x} dx$$

$$= \int_0^1 x^a dx$$

$$= \left[\frac{x^{a+1}}{a+1}\right]_0^1$$

$$I'(a) = \frac{1}{a+1}$$

Integrate w.r.t. a.

$$I(a) = \log(a+1) + c.$$

Substitute suitable value of a to find c

put $a = 0$

$$\therefore I(0) = \log 1 + c$$

but. $\int_0^1 \frac{x^0 - 1}{\log x} dx = \log 1 + c$

$$0 = 0 + c$$

$$\Rightarrow c = 0.$$

Hence: $\underline{I(a) = \log(a+1)}$

(2) Show that $\int_0^\infty e^{-(x^2 + a^2/x^2)} dx = \frac{\sqrt{\pi}}{2} e^{-2a}$

Sol^n:- Let $I(a) = \int_0^\infty e^{-\left(x^2 + \frac{a^2}{x^2}\right)} dx$ ——(1)

$$I'(a) = \int_0^\infty \frac{\partial}{\partial a}\left[e^{-\left(x^2 + \frac{a^2}{x^2}\right)}\right] dx$$

$$= \int_0^\infty e^{-\left(x^2 + \frac{a^2}{x^2}\right)} \cdot \left(-\frac{2a}{x^2}\right) dx.$$

put $\frac{a}{x} = t$ $\therefore \frac{-a}{x^2} dx = dt$

x	0	∞
t	∞	0

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$$I'(a) = \int_{\infty}^{0} e^{-\left(\frac{a^2}{t^2} + t^2\right)} \cdot 2\,dt$$

$$= -2 \int_{0}^{\infty} e^{-\left(t^2 + \frac{a^2}{t^2}\right)} dt$$

$I'(a) = -2\, I(a)$ ... from eqⁿ ①

$\therefore \frac{I'(a)}{I(a)} = -2$

Integrate w.r.t a.

$\log I(a) = -2a + c$

$\therefore I(a) = e^{-2a + c}$

$\therefore I(a) = e^{-2a} \cdot e^{c}$

Let $e^{c} = A \Rightarrow I(a) = A \cdot e^{-2a}$ ——— (2)

put $a = 0$ in $I(0) = A$.

$\therefore$ eqⁿ(1) $\Rightarrow \int_0^{\infty} e^{-x^2} dx = A$

$\therefore A = \frac{\sqrt{\pi}}{2}$

$\therefore$ from eqⁿ (2), $I(a) = \frac{\sqrt{\pi}}{2} e^{-2a}$.

Rule-(II) Integral with limits as functions of the parameter (Leibnitz Rule).

If $I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)\, dx$, where a and b are functions of the parameter α, then

$$\frac{dI}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial \alpha} \{f(x,\alpha)\} dx + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}.$$

EX. ① show that $\int_{\pi/6a}^{\pi/2a} \frac{\sin ax}{x} dx$ is independent of a.

Solⁿ:- To show that $I'(a) = 0$.

$$I'(a) = \frac{d}{da} \int_{\pi/6a}^{\pi/2a} \frac{\sin ax}{x} dx$$

Apply DUIS. $I'(a) = \int_{\pi/6a}^{\pi/2a} \frac{\partial}{\partial a}\left(\frac{\sin ax}{x}\right) dx + \left\{\frac{d}{da}\left(\frac{\pi}{2a}\right)\right\} \frac{\sin a\frac{\pi}{2a}}{\left(\frac{\pi}{2a}\right)}$

$$- \left\{\frac{d}{da}\left(\frac{\pi}{6a}\right)\right\} \frac{\sin\left(a\frac{\pi}{6a}\right)}{\left(\frac{\pi}{6a}\right)}$$

$$= \int_{\pi/6a}^{\pi/2a} \frac{\cos ax . x \, dx}{x} + \left(\frac{-\pi}{2a^2}\right)\frac{1}{(\pi/2a)} - \left(\frac{-\pi}{6a^2}\right)\frac{(1/2)}{(\pi/6a)}.$$

$$= \left[\frac{\sin ax}{x}\right]_{\pi/6a}^{\pi/2a} - \frac{1}{a} + \frac{1}{2a}$$

$$= \frac{1}{a}\left[\sin\frac{\pi}{2} - \sin\frac{\pi}{6}\right] - \frac{1}{a} + \frac{1}{2a}$$

$$= \frac{1}{a} - \frac{1}{2a} + \frac{1}{2a} - \frac{1}{2a} = 0.$$

$\therefore$ $I'(a) = 0$ implies that $I(a)$ is independent of a.

Que (2) If $y = \int_0^x f(t) \sin a(x-t)\,dt$, show that $\frac{d^2y}{dx^2} + a^2 y = a f(x)$

Sol^n:- Given $y = \int_0^x f(t) \sin a(x-t)\,dt$.

Differentiating by rule (II), w.r.t. x.

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x} f(t)\sin a(x-t)\,dt + \left\{\frac{d}{dx}x\right\} f(x)\sin a(x-x) - \left\{\frac{d}{dx}(0)\right\} f(0).\sin 0.$$

$$= \int_0^x a.f(t)\cos a(x-t)\,dt + 0 + 0.$$

Again, differentiating w.r.t. x

$$\frac{d^2y}{dx^2} = \int_0^x \frac{\partial}{\partial x}\left[a f(t)\cos a(x-t)\right]dt + \left[\frac{d}{dx}(x)\right] a f(x)\cos 0 - \frac{d}{dx}(0).a f(0)\cos 0$$

$$= \int_0^x a\, f(t) \cdot (-\sin a(x-t)) \cdot a\, dt + a \cdot f(x) - 0$$

$$= -a^2 \int_0^x f(t) \sin a(x-t)\, dt + a f(x)$$

$$\frac{d^2y}{dx^2} = -a^2 y + a f(x)$$

$$\therefore \frac{d^2y}{dx^2} + a^2 y = a f(x)$$

ERROR FUNCTION:-

(1) **Def^n:-** Error function of x or probability integral defined as the integral.

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

(2) Complementary Error Function:-

Complementary Error function x is defined as.

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du$$

(3) Alternate definition of Error function:-

In integral of (1), if we put $u^2 = t$.

$\therefore u^2 = t \Rightarrow 2u\, du = dt$

$$du = \frac{dt}{2\sqrt{t}}$$

u	0	x
t	0	x^2

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^{x^2} e^{-t} \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{x^2} e^{-t}\, t^{-1/2}\, dt$$

$$\therefore \boxed{\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_0^{x^2} e^{-t} \cdot t^{-1/2}\, dt}$$

This is also considered as definition of Error function of x.

Properties of Error Functions:-

(1) erf(∞) :-

$$erf(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du$$

(put $u^2 = y$)
$\Rightarrow du = \frac{1}{2} t^{-1/2} dt$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-y} \cdot \frac{1}{2} y^{-1/2} dy$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y} \cdot y^{-1/2} dy$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\frac{1}{2} = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1$$

$$\therefore \boxed{erf(\infty) = 1}$$

(2) erf(0) =

$$erf(0) = \frac{2}{\sqrt{\pi}} \int_0^0 e^{-u^2} du = 0$$

$$\boxed{\therefore erf(0) = 0}$$

(3) erf(x) + erfc(x) =

$$\therefore erf(x) + erfc(x) = \frac{2}{\sqrt{\pi}} \left[\int_0^x e^{-u^2} du + \int_x^\infty e^{-u^2} du \right]$$

$$= \frac{2}{\sqrt{\pi}} \left[\int_0^\infty e^{-u^2} du \right]$$

$$= erf(\infty)$$

$$\therefore \boxed{erf(x) + erfc(x) = 1}$$

(4) erf(x) is an odd function:-

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

Replace x by $-x$.

$$\therefore erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-u^2} du$$

put $u = -y \Rightarrow du = -dy$

u	0	$-x$
y	0	x

$$erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} (-dy)$$

$$= \frac{-2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$$

$\mathrm{erf}(-x) = -\mathrm{erf}(x)$ Thus $\mathrm{erf}(x)$ is an odd funct

(5) Expression for erf (x) in series:-

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

since $e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} \dots$

$$\therefore \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \left[1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \dots \right] du$$

$$= \frac{2}{\sqrt{\pi}} \left[u - \frac{u^3}{3} + \frac{u^5}{10} - \frac{u^7}{42} + \dots \right]_0^x$$

$$\therefore \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \dots \right]$$

This series is uniformly convergent & hence erf(x) is a continuous funct^n of x. Values of erf (x) can be tabulated using above series.

(6) Alternate definition of complementary error function:-

By result (4), $\mathrm{erf}(\infty) = 1$

$$\therefore \mathrm{erf}(\infty) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t} \cdot t^{-1/2} dt = 1$$

This can be written as,

$$\frac{1}{\sqrt{\pi}} \left\{ \int_0^{x^2} e^{-t^2} t^{-1/2} dt + \int_{x^2}^\infty e^{-t} t^{-1/2} dt \right\} = 1$$

$$\Rightarrow \frac{1}{\sqrt{\pi}} \int_0^{x^2} e^{-t^2} t^{-1/2} dt + \frac{1}{\sqrt{\pi}} \int_{x^2}^\infty e^{-t} t^{-1/2} dt = 1 \quad \text{---} (1)$$

Here first integral on L.H.S of (1) is erf(x). and second integral $= \frac{1}{\sqrt{\pi}} \int_{x^2}^\infty e^{-t} t^{-1/2} dt$ is defined as complementary error function. x i.e. erfC(x).

$$\therefore \boxed{\mathrm{erfC}(x) = \frac{1}{\sqrt{\pi}} \int_{x^2}^\infty e^{-t} \cdot t^{-1/2} dt}$$

$\therefore$ from eq^n(1),

$\Rightarrow \mathrm{erf}(x) + \mathrm{erfc}(x) = 1$

Differentiation of Error Function:-

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

$$\therefore \operatorname{erf}(ax) = \frac{2}{\sqrt{\pi}} \int_0^{ax} e^{-u^2} du$$

Using 2nd rule of differentiation under the integral sign, noting that integration is w.r.t. u and diffn is to be carried out w.r.t. x:

$$\frac{d}{dx}(\operatorname{erf}(ax)) = \frac{2}{\sqrt{\pi}}\left[\int_0^{ax} \frac{\partial}{\partial x} e^{-u^2} du + \left\{\frac{d}{dx}(ax)\right\} e^{-a^2x^2} - \left\{\frac{d}{dx}(0)\right\} e^{-0}\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[0 + a \cdot e^{-a^2x^2} - 0\right]$$

$$\therefore \boxed{\frac{d}{dx}\operatorname{erf}(ax) = \frac{2a e^{-a^2x^2}}{\sqrt{\pi}}}$$

Integration of Error function:-

$$\int_0^t \operatorname{erf}(ax)\,dx = \int_0^t 1 \cdot \operatorname{erf}(ax)\,dx.$$

Integrating by parts treating unity as a second function & $\operatorname{erf}(ax)$ as first function.

$$= \Big[\operatorname{erf}(ax) \cdot x\Big]_0^t - \int_0^t \frac{d}{dx}\operatorname{erf}(ax) \cdot x\,dx$$

$$= t\operatorname{erf}(at) - 0 - \int_0^t \frac{2a e^{-a^2x^2}}{\sqrt{\pi}} x\,dx$$

$$= t\operatorname{erf}(at) + \frac{1}{\sqrt{\pi}} \cdot \frac{1}{a}\int_0^t e^{-a^2x^2}(-2a^2x\,dx)$$

$$= t \cdot \operatorname{erf}(at) + \frac{1}{a\sqrt{\pi}}\left[e^{-a^2x^2}\right]_0^t$$

$$= t \operatorname{erf}(at) + \frac{1}{a\sqrt{\pi}}\left(e^{-a^2t^2} - 1\right)$$

$$\int_0^t \operatorname{erf}(ax)\,dx = t\operatorname{erf}(at) + \frac{1}{a\sqrt{\pi}} e^{-a^2t^2} - \frac{1}{a\sqrt{\pi}}$$

Ex:- ① prove that $erfc(-x) + erfc(x) = 2$.

$\Rightarrow$ we know that $erf(x) + erfc(x) = 1$

replace x by $-x \Rightarrow erf(-x) + erfc(-x) = 1$

$\therefore -erf(x) + erfc(-x) = 1$

$\because [erf(-x) = -erf(x)]$

$\therefore erfc(-x) = 1 + erf(x)$

$erfc(-x) + erfc(x) = 1 + erf(x) + erfc(x)$

$= 1 + 1$

$\therefore erfc(-x) + erfc(x) = 2$.

(2) show that $\int_0^\infty e^{-x^2-2bx}\,dx = \frac{\sqrt{\pi}}{2} \cdot e^{b^2}[1 - erf(b)]$

$\Rightarrow$ $I = \int_0^\infty e^{-x^2-2bx}\,dx = \int_0^\infty e^{-x^2-2bx-b^2+b^2}\,dx$.

$= e^{b^2}\int_0^\infty e^{-(x+b)^2}\,dx$.

put $x+b = u$, $dx = du$

x	0	∞
u	b	∞

$I = e^{b^2}\int_b^\infty e^{-u^2}\,du = e^{b^2}\,\frac{\sqrt{\pi}}{2} \cdot \frac{2}{\sqrt{\pi}}\int_b^\infty e^{-u^2}\,du$

$= \frac{\sqrt{\pi}}{2}\,e^{b^2} \cdot erfc(b)$.

$I = \frac{\sqrt{\pi}}{2}\,e^{b^2}[1 - erf(b)]$

(3) If $\alpha(x) = \sqrt{\frac{2}{\pi}}\int_0^x e^{-t^2/2}\,dt$ show that $erf(x) = \alpha[x\sqrt{2}]$.

soln.- $\alpha(x) = \sqrt{\frac{2}{\pi}}\int_0^x e^{-t^2/2}\,dt$

$\therefore \alpha[x\sqrt{2}] = \sqrt{\frac{2}{\pi}}\int_0^{x\sqrt{2}} e^{-t^2/2}\,dt$

put $\frac{t^2}{2} = u^2$ $\Rightarrow t = \sqrt{2}u$ $dt = \sqrt{2}u$

t	0	$x\sqrt{2}$
u	0	x

$$\therefore \alpha[x\sqrt{2}] = \frac{\sqrt{2}}{\sqrt{\pi}}\int_0^x e^{-u^2}\sqrt{2}\,du$$

$$= \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du.$$

$$\therefore \underline{\underline{\alpha[x\sqrt{2}] = \operatorname{erf}(x)}}$$